Resolutions for resonances

July 5, 2010

Abstract

1 Variance-covariance matrix

Given a vector of random variables:

$$\mathbf{X} = \begin{bmatrix} cX_1 \\ \vdots \\ X_n \end{bmatrix} ,$$

which is zero for independent variables (two variables are independent if they are not correlated).
The covariance matrix Σ is defined as the matrix whose (i, j) components are the covariances:

$$\Sigma_{ij} = \mathrm{cov}(X_i, X_j) = \mathrm{E}[(X_i - \mu_i)(X_j - \mu_j)] ,$$

where

$$\mu_i = E(X_i) ,$$

is the expected value of the X_i variable. That is, for a discrete random variable X with probability mass function $p(x)$:

$$\mathrm{E}(X) = \sum_i x_i p(x_i) ,$$

of course, for the continous case it is an integral.
Note that for $i = j$ the covariances become the variances, as $\sum_{ii} = \mathrm{E}[(X_i - \mu_i)^2] = \mathrm{var}[X_i]$.
The variance-covariance matrix thus has the form

$$\Sigma = \begin{bmatrix} \mathrm{E}[(X_1 - \mu_1)(X_1 - \mu_1)] & \mathrm{E}[(X_1 - \mu_1)(X_2 - \mu_2)] & \cdots & \mathrm{E}[(X_1 - \mu_1)(X_n - \mu_n)] \\ \mathrm{E}[(X_2 - \mu_2)(X_1 - \mu_1)] & \mathrm{E}[(X_2 - \mu_2)(X_2 - \mu_2)] & \cdots & \mathrm{E}[(X_2 - \mu_2)(X_n - \mu_n)] \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{E}[(X_n - \mu_n)(X_1 - \mu_1)] & \mathrm{E}[(X_n - \mu_n)(X_2 - \mu_2)] & \cdots & \mathrm{E}[(X_n - \mu_n)(X_n - \mu_n)] \end{bmatrix} .$$

It is a positive semi-definite symmetric matrix.

2 Mass resolution

In our case the function whose variance we want to compute is the resonance mass, which is a function of the kinematic variables of the two muons $\mathrm{M}(p_{T_1}, \mathrm{cotg}(\theta_1), \phi_1, p_{T_2}, \mathrm{cotg}(\theta_2), \phi_2)$ and these are correlated variables. We should than compute the variance of M from standard error propagation:

$$\begin{aligned}
\mathrm{var(M)} &= \sum_{ij} \frac{\partial M}{\partial X_i} \frac{\partial M}{\partial X_j} = \\
&= \left(\frac{\partial M}{\partial p_{T_1}}\right)^2 \mathrm{var}(p_{T_1})^2 + \left(\frac{\partial M}{\partial \cot(\theta_1)}\right)^2 \mathrm{var}(\cot(\theta_1))^2 + \\
&+ 2\left(\frac{\partial M}{\partial p_{T_1}}\right)\left(\frac{\partial M}{\partial \cot(\theta_1)}\right) \mathrm{cov}(p_{T_1}, \cot(\theta_1)) + \cdots .
\end{aligned} \tag{1}$$

The variance-covariance matrix in our case has more symmetries that can be exploited:

$$\Sigma = \begin{bmatrix}
\mathrm{var}(p_{T_1}) & \mathrm{cov}(p_{T_1}, \cot(\theta_1)) & \mathrm{cov}(p_{T_1}, \phi_1) & \mathrm{cov}(p_{T_1}, \phi_2) & \mathrm{cov}(p_{T_1}, \cot(\theta_2)) & \mathrm{cov}(p_{T_1}, p_{T_2}) \\
\vdots & \mathrm{var}(\cot(\theta_1)) & \mathrm{cov}(\cot(\theta_1), \phi_1) & \mathrm{cov}(\cot(\theta_1), \phi_2) & \mathrm{cov}(\cot(\theta_1), \cot(\theta_2)) & \mathrm{cov}(\cot(\theta_1), p_{T_2}) \\
 & \vdots & \mathrm{var}(\phi_1) & \mathrm{cov}(\phi_1, \phi_2) & \mathrm{cov}(\phi_1, \cot(\theta_2)) & \mathrm{cov}(\phi_1), p_{T_2}) \\
 & & \vdots & \mathrm{var}(\phi_2) & \mathrm{cov}(\phi_2, \cot(\theta_2)) & \mathrm{cov}(\phi_2), p_{T_2}) \\
 & & & \vdots & \mathrm{var}(\cot(\theta_2)) & \mathrm{cov}(\cot(\theta_2), p_{T_2}) \\
 & & & & \vdots & \mathrm{var}(p_{T_2})
\end{bmatrix},$$

where we omitted writing the symmetric part. A specific symmetry we can use comes from the fact that the two muons are undistiguishable (note that we are not taking into account the charge and in the code, but in the code μ_1 is always the minus charged muon), that is we can exchange index 1 with 2 and obtain the same result.
This means that the covariance terms opposite with respect to the anti-diagonal are the same (e.g. $\mathrm{cov}(\cot(\theta_1), p_{T_1}) = \mathrm{cov}(\cot(\theta_2), p_{T_2})$, and also $\mathrm{cov}(\cot(\theta_1), p_{T_2}) = \mathrm{cov}(\cot(\theta_2), p_{T_1})$.
Furthermore, the variances of single muon quantities depend only on the detector and algorithms used, not on the resonance from which they decayed, therefore $\mathrm{var}(p_{T_1}) = \mathrm{var}(p_{T_2})$, $\mathrm{var}(\cot(\theta_1) = \mathrm{var}(\cot(\theta_2))$ and $\mathrm{var}(\phi_1) = \mathrm{var}(\phi_2)$.
With these simplifications the matrix becomes

$$\Sigma = \begin{bmatrix}
\mathrm{var}(p_{T_1}) & \mathrm{cov}(p_{T_1}, \cot(\theta_1)) & \mathrm{cov}(p_{T_1}, \phi_1) & \mathrm{cov}(p_{T_1}, \phi_2) & \mathrm{cov}(p_{T_1}, \cot(\theta_2)) & \mathrm{cov}(p_{T_1}, p_{T_2}) \\
\vdots & \mathrm{var}(\cot(\theta_1)) & \mathrm{cov}(\cot(\theta_1), \phi_1) & \mathrm{cov}(\cot(\theta_1), \phi_2) & \mathrm{cov}(\cot(\theta_1), \cot(\theta_2)) & \cdots \\
 & \vdots & \mathrm{var}(\phi_1) & \mathrm{cov}(\phi_1, \phi_2) & \cdots & \\
 & & \vdots & \cdots & \cdots & \\
 & & & \vdots & \cdots & \\
 & & & & \vdots & \cdots
\end{bmatrix},$$

where, as before, only the independent components are written. So ultimately, the independent components we need to compute are:

$$\mathrm{var}(p_T),\ \mathrm{var}(\cot(\theta)),\ \mathrm{var}(\phi),$$

$$\mathrm{cov}(p_T, \cot(\theta)),\ \mathrm{cov}(p_T, \phi),\ \mathrm{cov}(\cot(\theta), \phi),$$

$$\mathrm{cov}(p_{T_1}, p_{T_2}),\ \mathrm{cov}(\cot(\theta_1), \cot(\theta_2)),\ \mathrm{cov}(\phi_1, \phi_2),$$
$$\mathrm{cov}(p_{T_{(1,2)}}, \cot(\theta_{(2,1)})),\ \mathrm{cov}(p_{T_{(1,2)}}, \phi_{(2,1)})),\ \mathrm{cov}(\cot(\theta_{(1,2)}), \phi_{(2,1)})\ ,$$

where line two refers to same-muon quantities and the indeces in line four indicate that we must use both permutations. Indeed, since we take μ_1 undistiguishable from μ_2, we can use both muons to compute the same variance or covariance.

The covariances are specific for each resonance.

3 Parameters reduction

Using the full set of twelve independent functions needed to express the mass resolution as a function of muon kinematics would lead to an unacceptably large number of parameters. One approach can be to look at the contribution of every single term in the variance expression,

$$\left(\frac{\partial M}{\partial X_i}\right)\left(\frac{\partial M}{\partial X_j}\right)\mathrm{cov}(X_i, X_j)$$

to determine the negligible terms and use only the most important ones.

It would also be possible to insert the full equation and unlock the parameters of lower order terms only in successive iterations.

3.1 Terms comparison

The covariance terms depend on the resonance. We consider here a sample of 10.000 Υ. All the results shown here refer to this sample and we will need to repeat this study for each resonace. We show in figures **??** the error function terms obtained multiplying the partial derivatives:

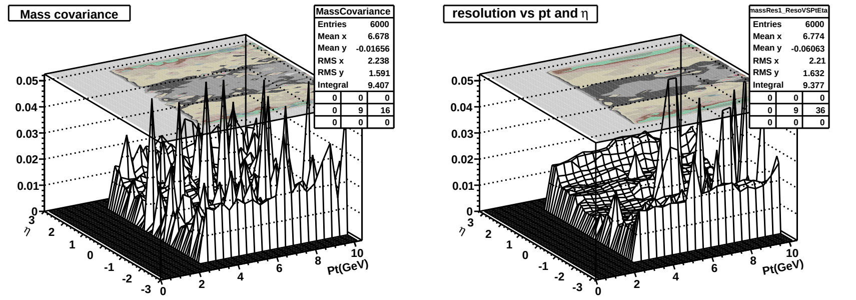
$$\frac{\partial M}{\partial p_{T_1}} = \left(\frac{p_{T_1}}{\sin(\theta_1)^2}\sqrt{\frac{\left(\frac{p_{T_2}}{\sin(\theta_2)}\right)^2 + M_\mu^2}{\left(\frac{p_{T_1}}{\sin(\theta_1)}\right)^2 + M_\mu^2}} - p_{T_2}\left(\cos(\phi_1 - \phi_2) + \frac{\cos(\theta_1)\cos(\theta_2)}{\sin(\theta_1)\sin(\theta_2)}\right)\right)/M\ , \tag{2}$$

$$\frac{\partial M}{\partial \phi_1} = p_{T_1} p_{T_2}\sin(\phi_1 - \phi_2)/M\ , \tag{3}$$

$$\frac{\partial M}{\partial \cot(\theta_1)} = \left(p_{T_1}^2\frac{\cos(\theta_1)}{\sin(\theta_1)}\sqrt{\frac{\left(\frac{p_{T_2}}{\sin(\theta_2)}\right)^2 + M_\mu^2}{\left(\frac{p_{T_1}}{\sin(\theta_1)}\right)^2 + M_\mu^2}} - p_{T_1}p_{T_2}\left(\frac{\cos(\theta_2)}{\sin(\theta_2)}\right)\right)/M\ , \tag{4}$$

by the corresponding covariance term. All the terms which have the same covariance are put together and the appropriate coefficient is used.

We compare the σ_M obtained from MC comparison with the one we get from the error propagation by including only the p_T term. The inclusion of other terms does not change the result significantly.



(a) Mass resolution from comparison with MC.

(b) Mass resolution from error propagation using only the p_T term.

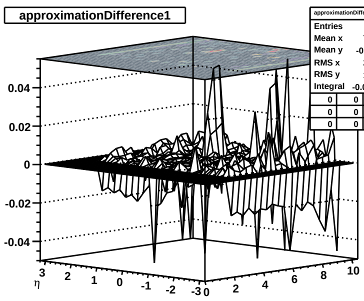
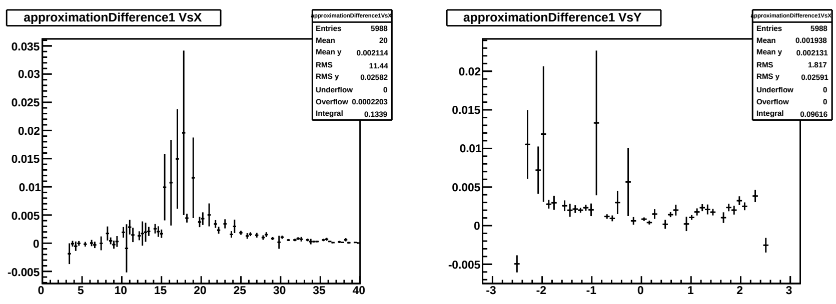


Figure 1: Difference of the resolutions in figures 2(c) and 2(d).

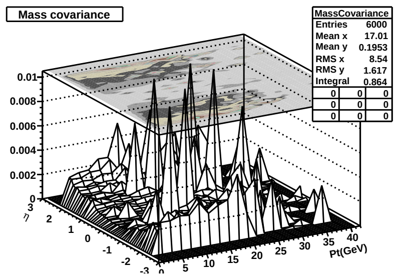


(a) Resolutions difference vs p_T.

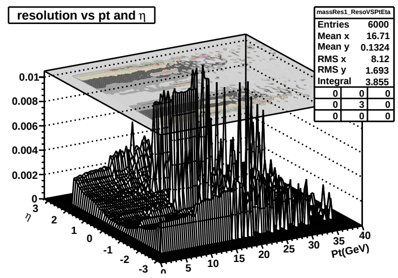
(b) Resolutions difference vs η.

3.2 J/Ψ sample

We repeat the resolution check on the full sample of J/Ψ events at our disposal. The results are reported in figure **??**



(c) Mass resolution from comparison with MC.



(d) Mass resolution from error propagation using only the p_T term.

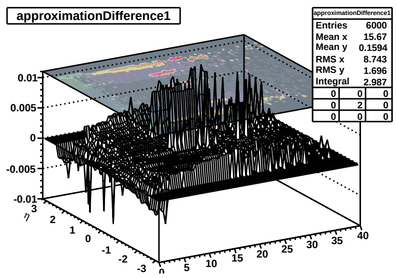
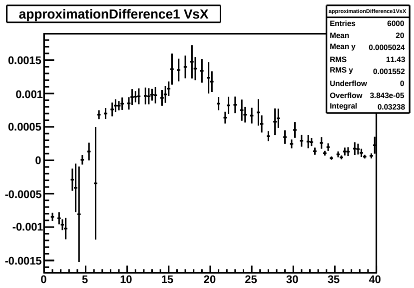


Figure 2: Difference of the resolutions in figures 2(c) and 2(d).

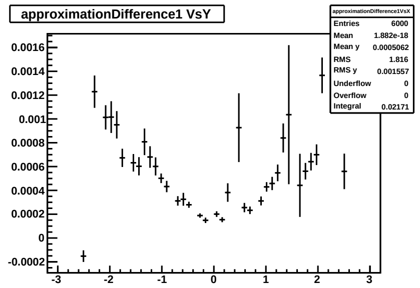
4 Functional form for resolution fit

In the previous section we found that the dominant term in the error propagation expression was the term of σp_T. We want to determine a functional form for the $\sigma p_T/p_T$ as a function of p_T and η. This will allow the fit to extract the parameters of the resolution on the p_T of the muons from the expression of the resolution on the mass of the resonance.

In order to derive this expression we want to factorize the dependence on p_T and η, fit them separately and then put them together. Using simple 1d fits we have much more control on the convergence. The resolution on the p_T of a muon only depends on the characteristics of the detector and on reconstruction algorithms, so in principle muons from any source could be used to extract the functional form. However, the statistics at our disposal are limited and the correlations between muons from resonances lead to having a big bias to the distribution of $\sigma p_T/p_T$ that we can derive from those samples. To avoid this problem we simulated 100.000 muons from a muon gun source with a flat

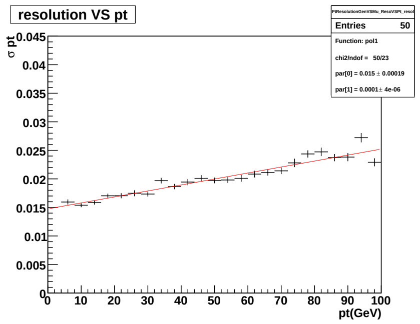


(a) Resolutions difference vs p_T.

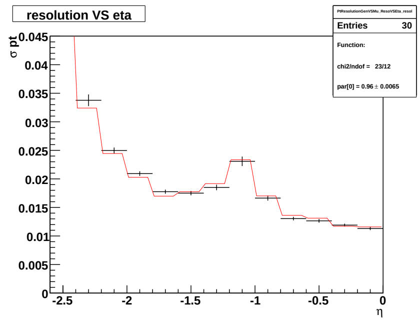


(b) Resolutions difference vs η.

spectrum in $p_T \in [5, 100]$ and $\eta \in [-3, 3]$. Since these muons are really (to a good approximation) uncorrelated we can factorize and fit the dependences separately. The distributions of $\sigma p_T/p_T$ vs p_T and η are shown in figures 3(c) and 3(d) respectively. The $\sigma p_T/p_T$ is fitted with a function of the form $par[0] +$



(c) $\sigma p_T/p_T$ vs p_T. The function fitted is a $1/p_T$ dependence.



(d) $\sigma p_T/p_T$ vs η. To better describe the spikes at $\eta \approx 1$ we used a per-point function.

$par[1]/p_T$ and the values of the parameters are also shown in the figure. For the η dependence we used a per-point function with a different value in each bin, in order to better describe the spikes at $\eta \approx 1$. In this case the only free parameter is the overall scale of the function $par[0]$ which again is reported in the corresponding figure.

Deriving the values of $\sigma p_T/p_T$ vs p_T and η from the projections we get: $\sigma p_T/p_T(p_T) + c_1$ and $\sigma p_T/p_T(\eta) + c_2$. c_1 and c_2 contain the mean contribution from η and p_T respectively. Since we were using a muon gun, with muons distribution isotropic in p_T and η, the mean value in p_T does not depend on η and vice versa. This allowed us to factorize the dependence. When we put the two functions together though, we get: $\sigma p_T/p_T(p_T, \eta) = \sigma p_T/p_T(p_T) + \sigma p_T/p_T(\eta) + c_1 + c_2$, which means a double counting of the contributions. From the factorization we deterimined the shape of the functional form, but not the scale. To deterimine the scale factor $c \neq c_1 + c_2$ we require the $\sigma p_T/p_T(p_T, \eta)$ function for a given point. We use the distribution of the variance $\text{var}_{p_T}(p_T, \eta)$ and chose the bin with $p_T = 5$ and $\eta = 0$ because was in a region with good statistics. Since we are interested in finding a starting value for the fit and not a very precise function we can accept to have a big uncertainty on the scale factor.

In the following we report the values used for the fit.

For η by-points the 0-0.2 bin is 0.00942984 and the parameters of $\sigma p_T/p_T(p_T)$ are: $par[0] = 1.205793e-02, par[1] = 2.047489e-04$. For $p_T = 5$ we get 0.0131, thus $\sigma p_T/p_T(p_T = 5, \eta = 0) = 0.0094 + 0.0131 = 0.0225$ The value of this bin from the $\sqrt{var(p_T)} = 0.00720$. Requiring the equality of the two values we derive the value we need to sum to the constant to be $c = -0.015$. The scale of the resolution function changes from $par[0]$ to $par[0] + c = 0.012 - 0.015 = -0.003$.